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Hardness of computing quantum invariants of 3-manifolds with restricted topology

2025 DataShape Workshop

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The **Reshetikhin-Turaev (RT)** is a family defined for **both** 3-manifolds and knots

Theorem (Kuperberg, 2009; Algic and Lo, 2014):
"Computing (or even approximating within good accuracy) some choices of the RT invariant is #P-hard"

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$$\mathtt{P} \stackrel{?}{=} \mathtt{NP}$$

- #P-hard: for a given term, **counting** how many assignments make the term $\mathtt{TRUE}$

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Does restricting the topology yields to easier algorithms?

- A manifold M is **irreducible*** if M is not homemorphic to the direct sum $N_1 \# N_2$ where $N_1, N_2 \neq S^3$
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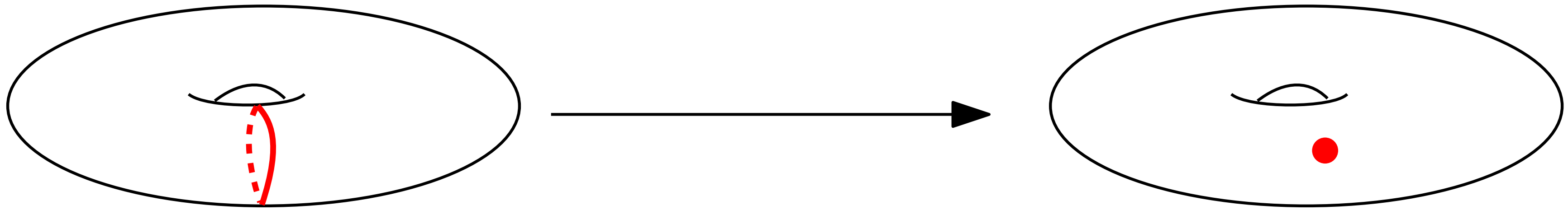

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Does restricting the topology yields to easier algorithms?

(H.E. and C.M., 2025) No

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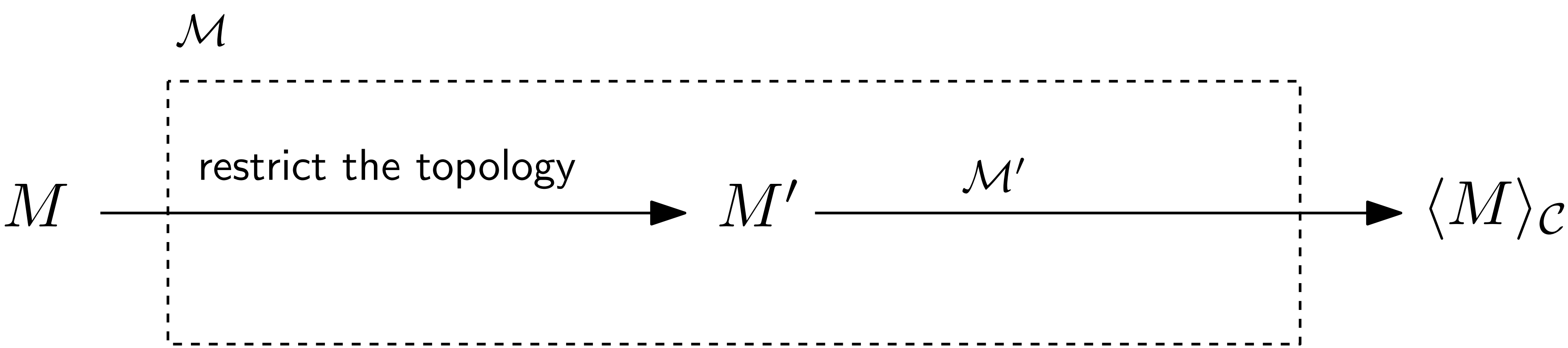

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$\mathcal{M}$ "$\in$"#P-hard

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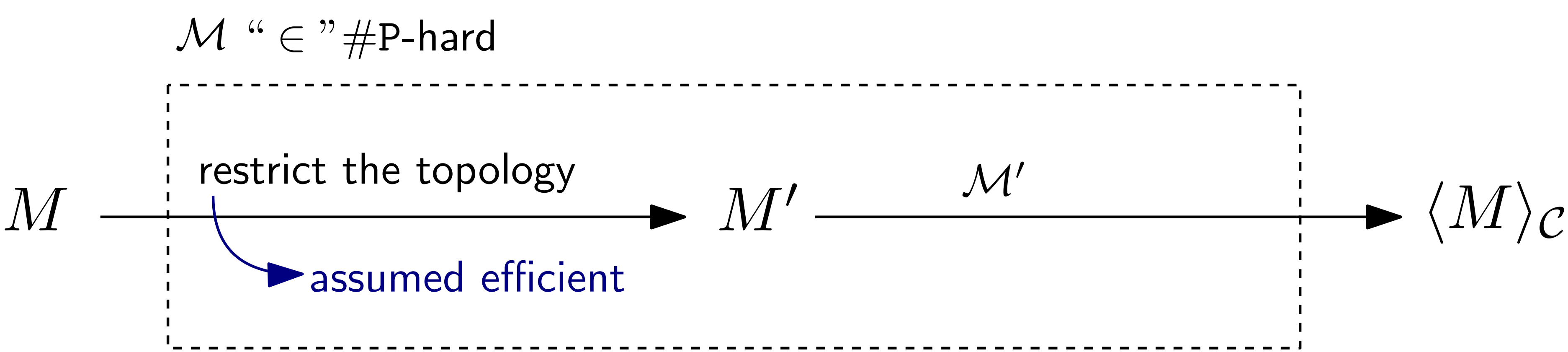

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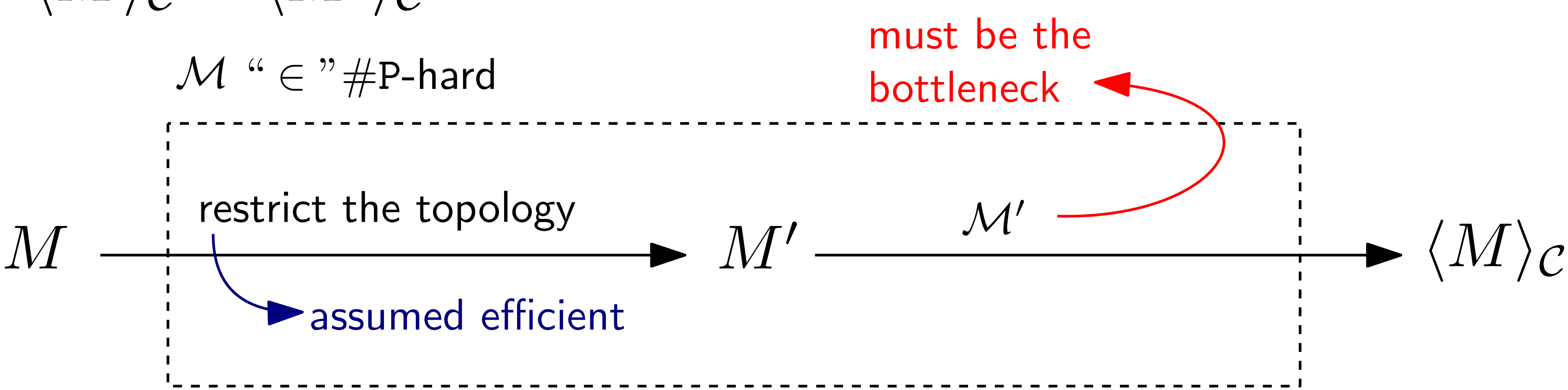

Computation of quantum invariants

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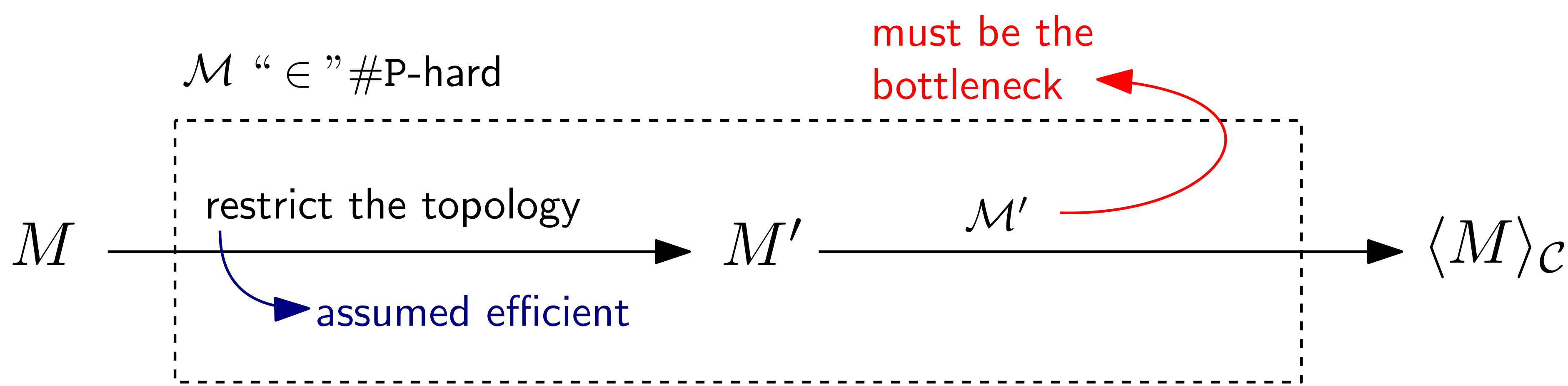
- Suppose we have a machine (**oracle**) $\mathcal{M}'$ that tells the invariant of any restricted manifold. We will use it to construct a machine $\mathcal{M}$ that tells the invariant for **any** manifold
- Show that we can change the manifold, in polynomial time, to a manifold with the same invariant



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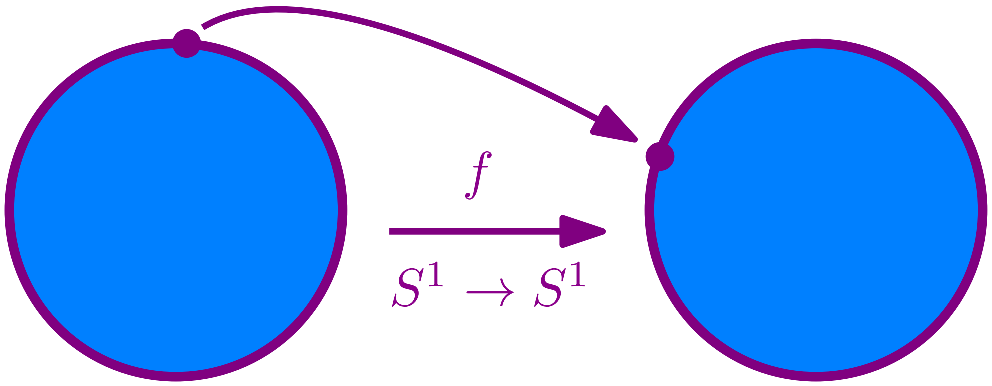

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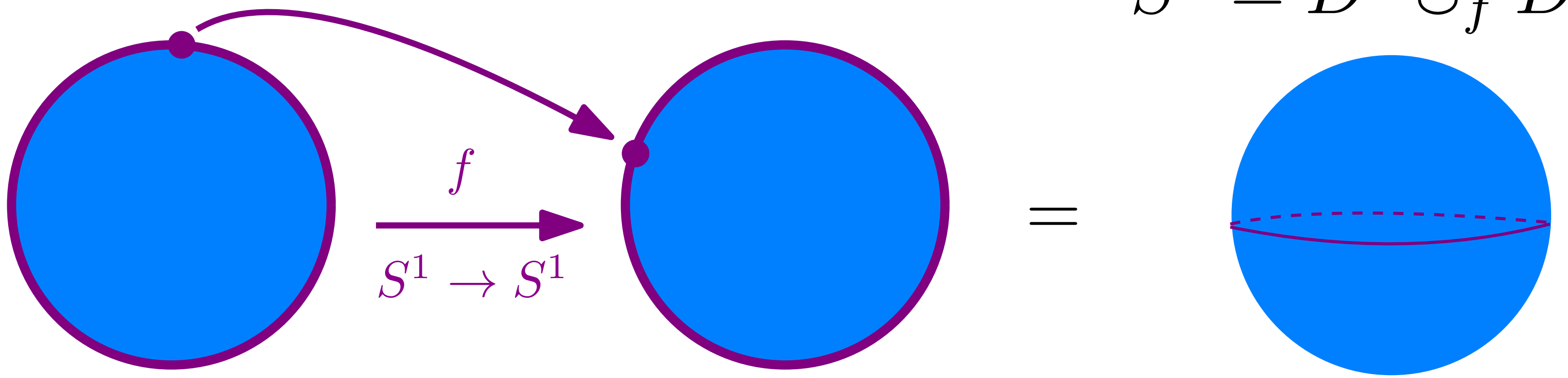
$$S^2 = D^2 \cup_f D^2$$



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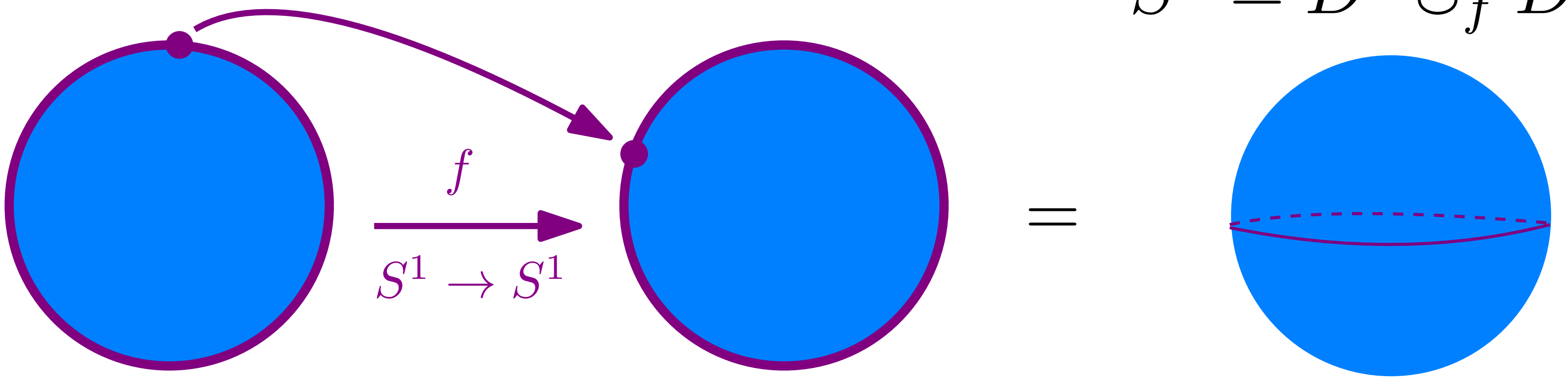
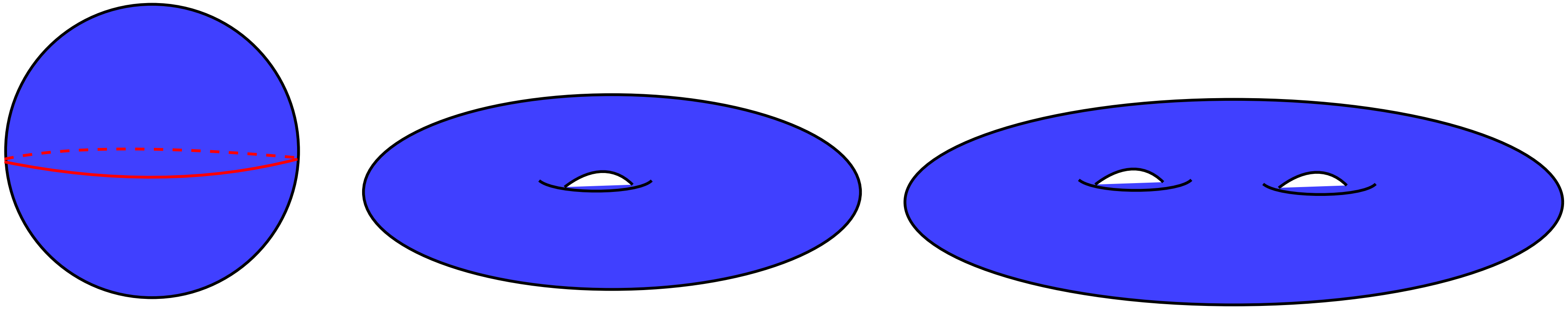


- **Handlebody**: a 3-manifold with boundary a closed surface



B^3 solid torus double solid torus

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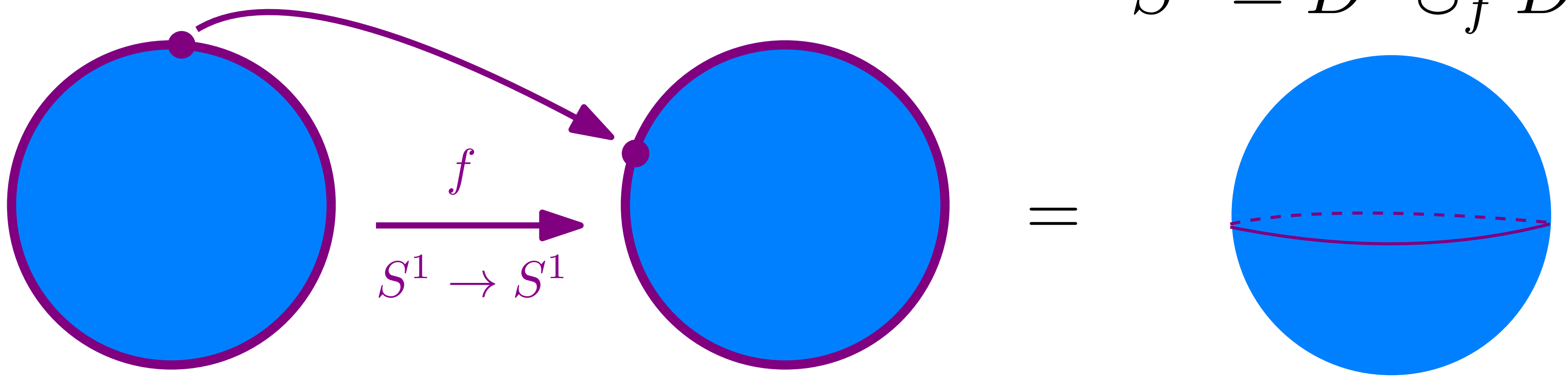

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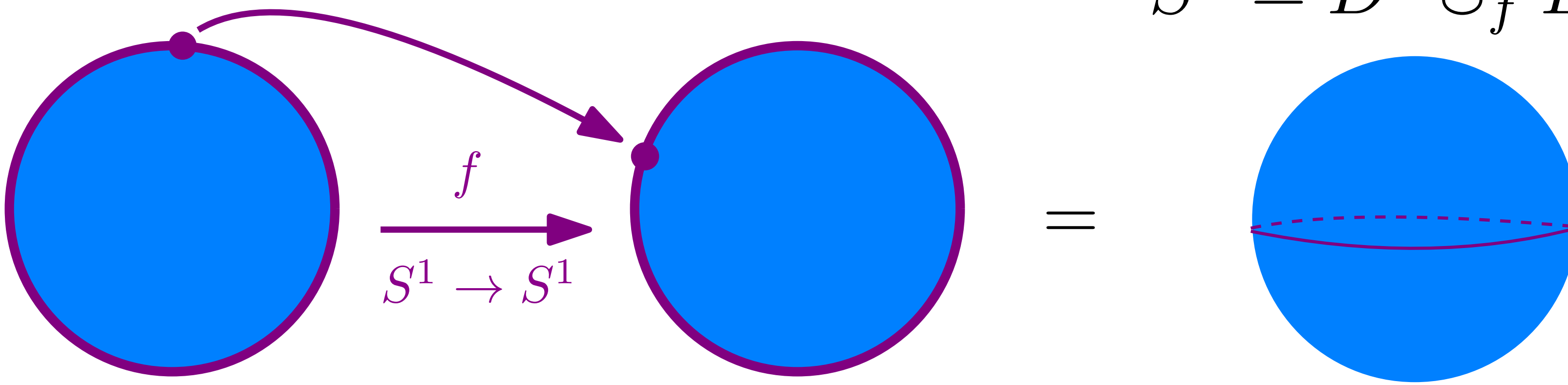

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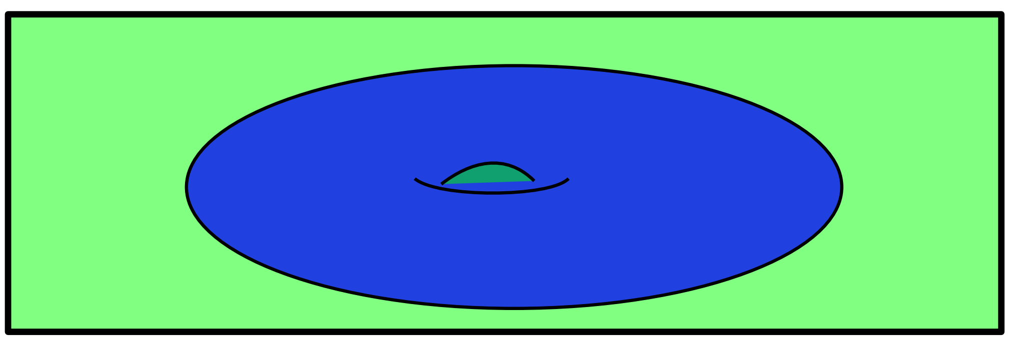

$S^3 = \mathbb{R}^3 \cup \{\infty\}$

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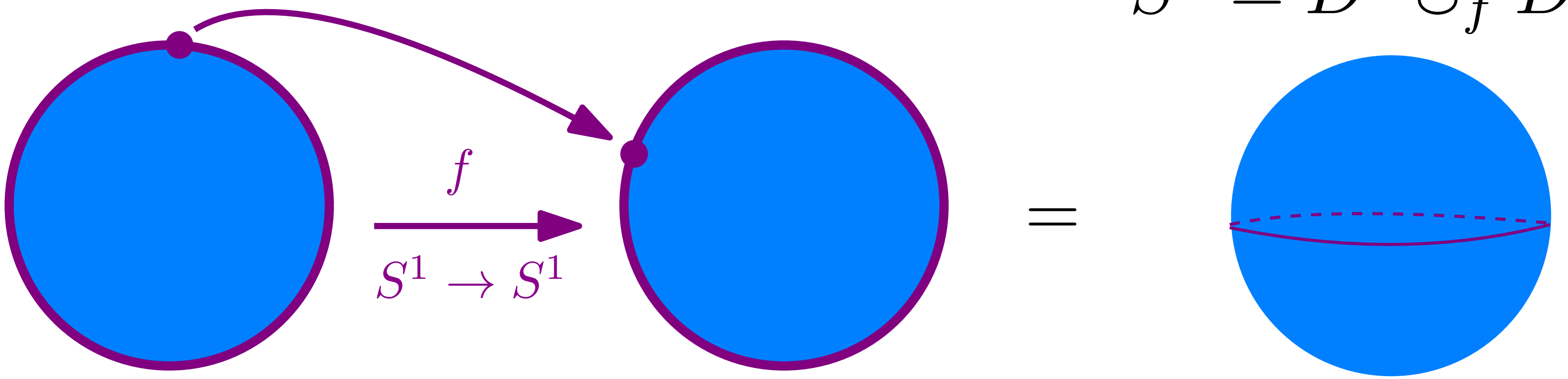

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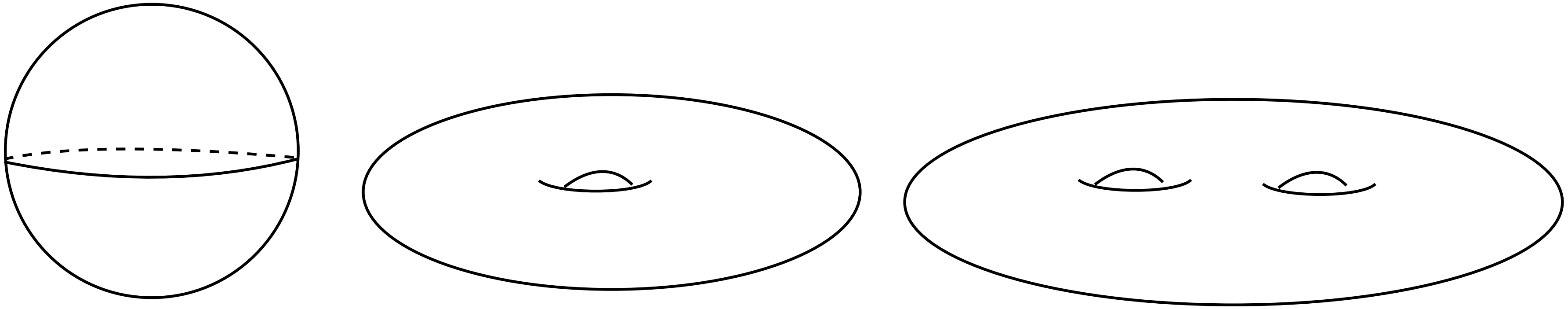
> "Every closed 3-manifold has a Heegaard splitting"

Surface theory

1. The **genus** of a *closed* surface is the number of copies of tori added to form the surface



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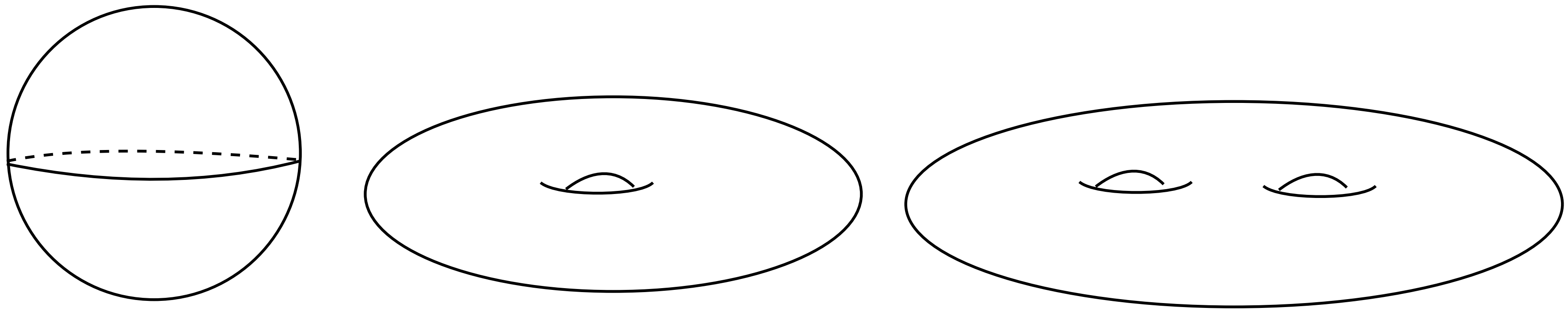

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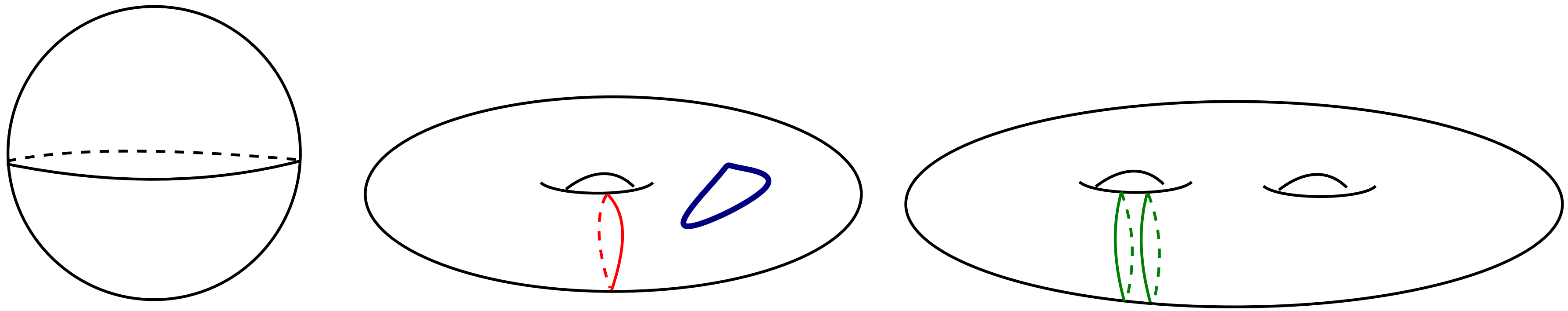

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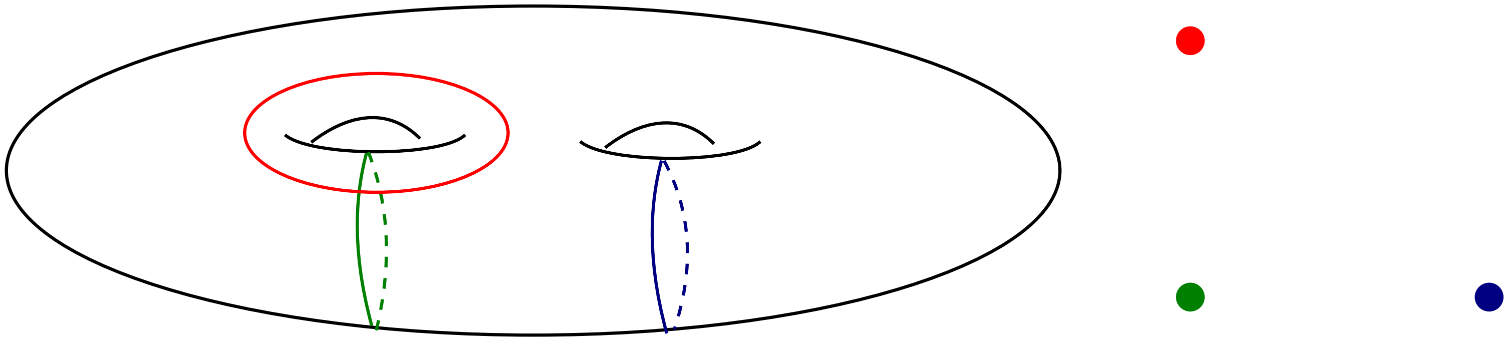
"Two closed surfaces Σ_g and $\Sigma_{g'}$ are homeomorphic if and only if $g = g'$"

2. An (essential) curve is (the image of) a proper embedding S^1 in Σ that does not bound a disk

Surface theory

3. The **curve graph**, $C(\Sigma_g)$, of a Σ_g
   - has the free homotopy classes of curves in the surface as vertices



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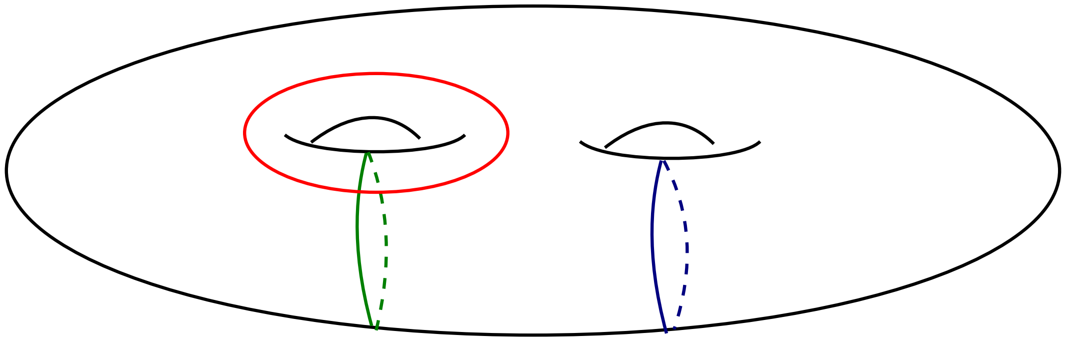

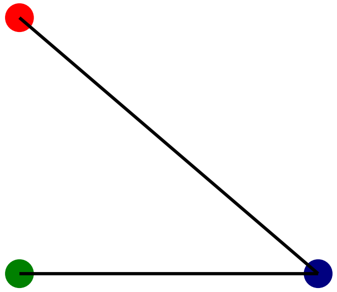

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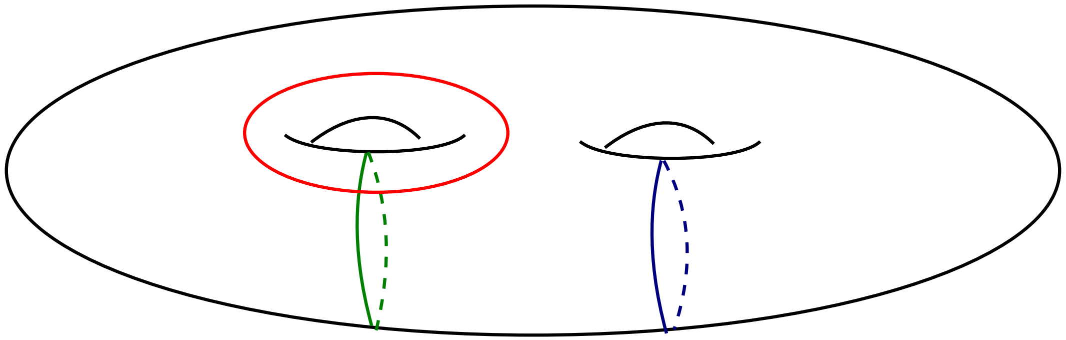

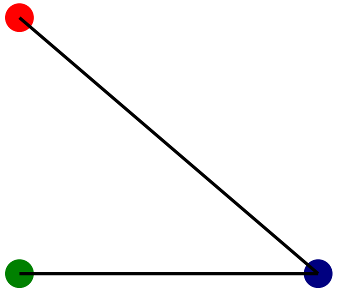

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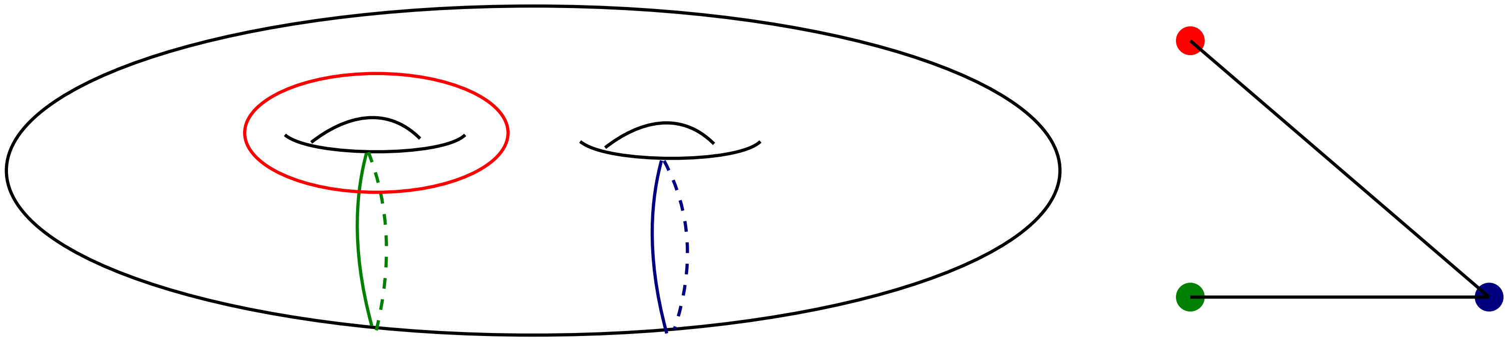

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The **mapping class group** of Σ_g

$$\mathrm{Mod}(\Sigma_g) = \mathrm{Homeo}^+(\Sigma_g)/\mathrm{Homotopies}$$

acts isometrically on the curve graph

Back to Heegaard splittings

Two copies of the handlebody $\mathcal{H}$ of boundary Σ_g can be glued by homeomorphism $f : \Sigma_g \to \Sigma_g$ to form a closed 3-manifold

$$M = \mathcal{H}_1 \cup_f \mathcal{H}_2$$

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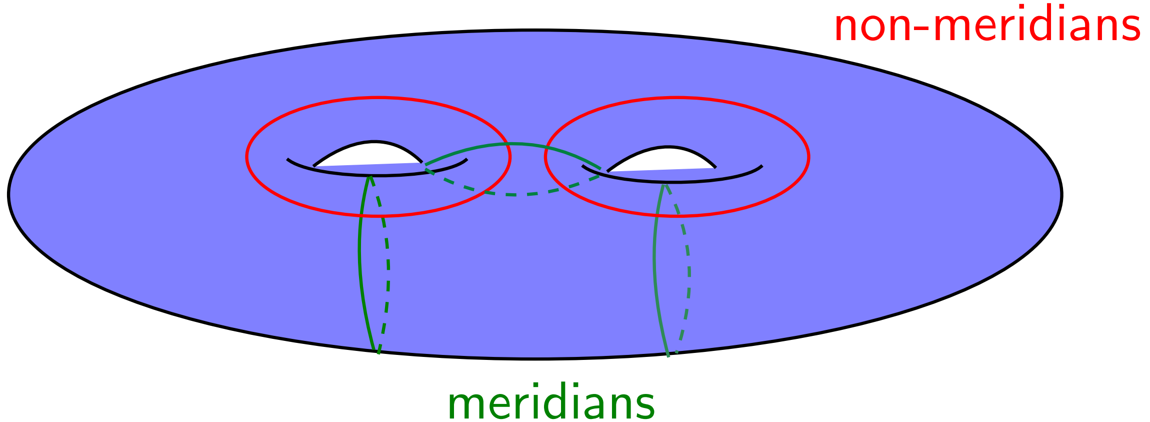

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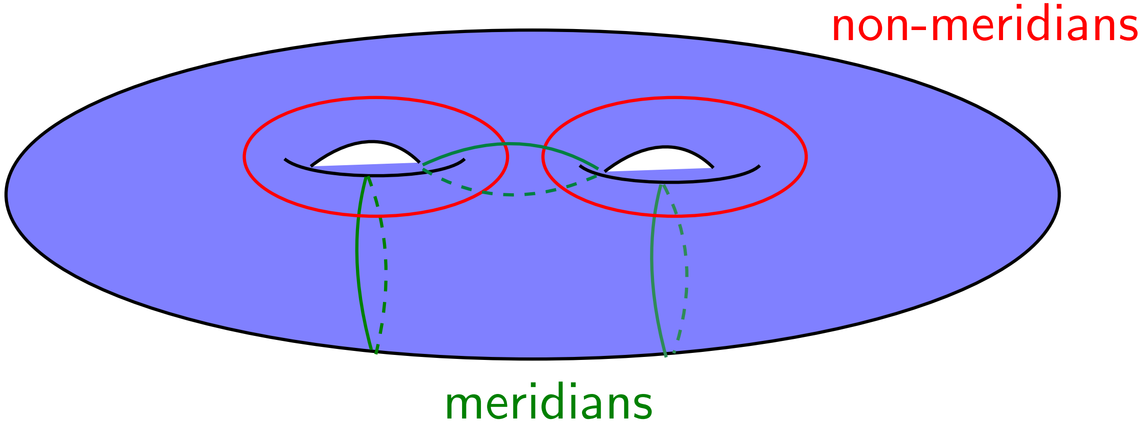

The **disk graph** $K(\mathcal{H})$ of a handlebody is the subgraph of $C(\partial\mathcal{H} = \Sigma_g)$ of all meridians

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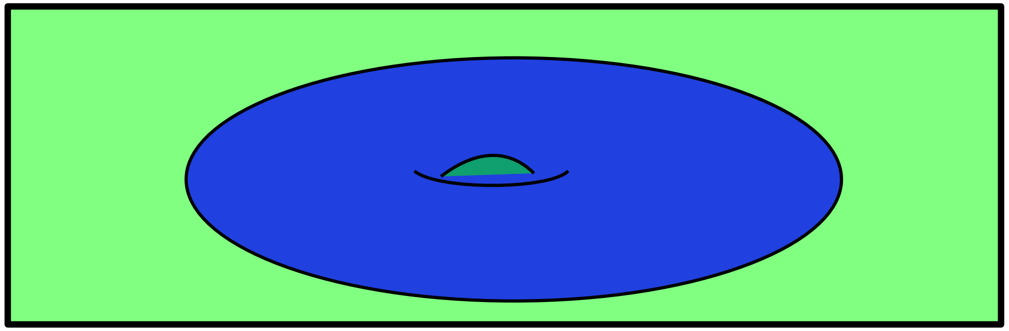

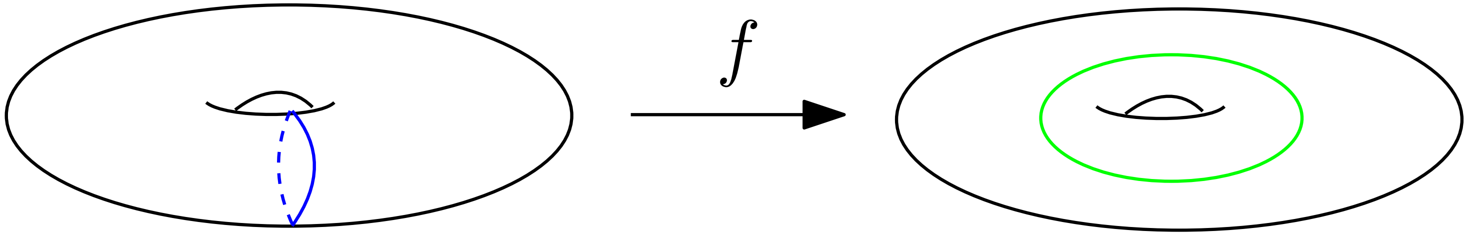

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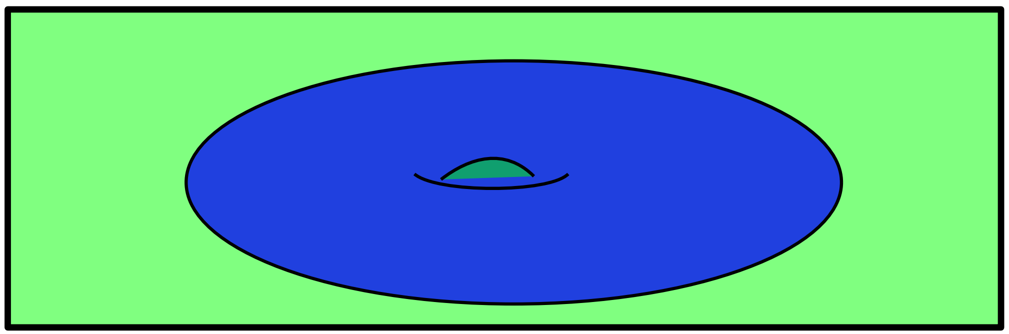

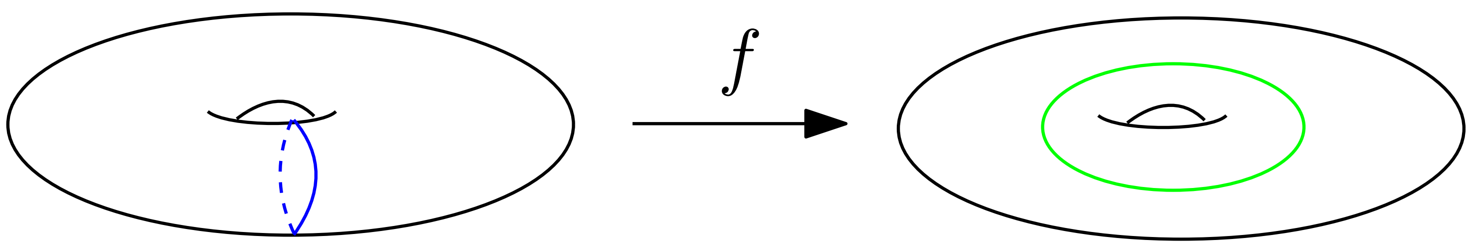

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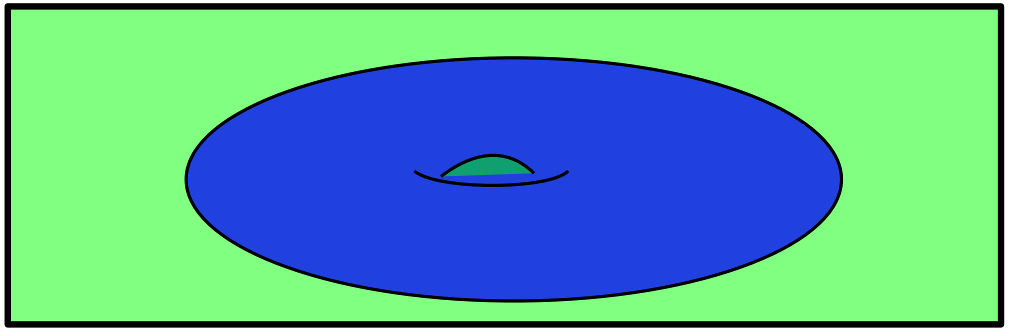

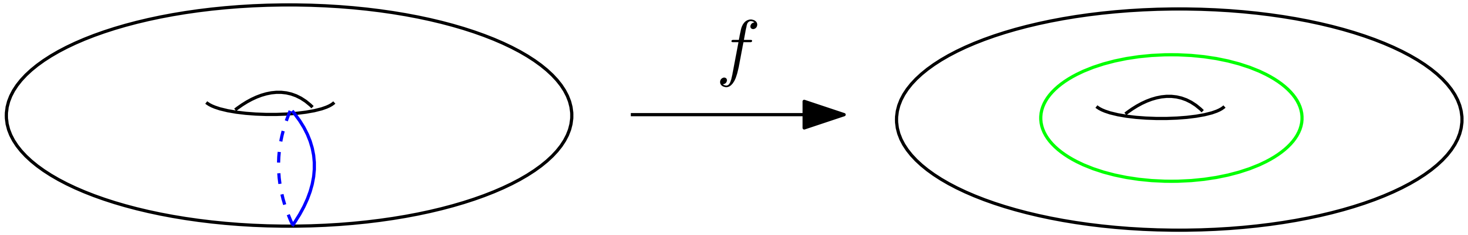

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The **Hempel distance** of the splitting $M = \mathcal{H}_1 \cup_f \mathcal{H}_2$ is $d_f := d(K(\mathcal{H}_1), K(\mathcal{H}_2))$

Our result

Theorem:

- if $d_f \geq 1$, the manifold is irreducible
- if $d_f \geq 3$, the manifold is hyperbolic
- if $d_f \geq k$, the manifold cannot embed an incompressible oriented surface of genus at least $2k$

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Theorem (Vafa,1988; Yoshizawa, 2014):

For every choice of RT invariant, there is a constant N (depending on k) and a map $\tau \in \mathrm{Mod}(\Sigma_g)$ such that

$$\langle \mathcal{H}_1 \cup_f \mathcal{H}_2 \rangle = \langle \mathcal{H}_1 \cup_{\tau^N \circ f} \mathcal{H}_2 \rangle \text{ and } d_{\tau^N \circ f} \geq k$$

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Given a Heegaard diagram of M a fixed choice of k, one can find in polynomial time (of degree $162 \times k^{1.6}$), a splitting of a 3-manifold M' that has the same RT-invariant of M, but of Hempel distance at least k

Thank you!